

Radon Numbers for Trees

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Abstract

Many interesting problems are obtained by attempting to generalize classical results on convexity in Euclidean spaces to other convexity spaces, in particular to convexity spaces on graphs. In this paper we consider P_3 -convexity on graphs. A set U of vertices in a graph G is P_3 -convex if every vertex not in U has at most one neighbour in U . More specifically, we consider Radon numbers for P_3 -convexity in trees.

Tverberg's theorem states that every set of $(k - 1)(d + 1) - 1$ points in \mathbb{R}^d can be partitioned into k sets with intersecting convex hulls. As a special case of Eckhoff's conjecture, we show that a similar result holds for P_3 -convexity in trees.

A set U of vertices in a graph G is *free*, if no vertex of G has more than one neighbour in U . We prove an inequality relating the Radon number for P_3 -convexity in trees with the size of a maximal free set.

1 Introduction

Radon's classical lemma [8] states that every set of $d + 2$ points in \mathbb{R}^d can be partitioned into two sets whose convex hulls intersect. Tverberg [9] generalized this to partitions into more than two sets. Namely, every set of at least $(k - 1)(d + 1) + 1$ points in \mathbb{R}^d can be partitioned into k sets whose convex hulls have a point in common.

Inspired by this, Eckhoff conjectured in [3] that the situation is similar in general convexity spaces. A *convexity space* is a pair (X, \mathcal{C}) where X is a set and \mathcal{C} is a collection of subsets of X , called *convex sets*, such that \emptyset and X are convex and the intersection of convex sets is convex. The *convex hull* of a set $S \subseteq X$, denoted by $H_{\mathcal{C}}(S)$, is the minimal convex set containing S , i.e. the intersection of all convex sets containing S . For a set S , a k -Radon partition is a partition of S into k sets whose convex hulls have a point in common. A set is k -anti Radon (or k -a.r.) if it has no k -Radon partition. The k th Radon number of (X, \mathcal{C}) is the minimal number (if it exists) $r_k(\mathcal{C})$ such that every set $S \subseteq X$ of size at least $r_k(\mathcal{C})$ has a k -Radon partition. Eckhoff [3] conjectured that $r_k(\mathcal{C}) \leq (k - 1)(r_2(\mathcal{C}) - 1) + 1$ holds in every convexity space. This conjecture has been proved in several convexity spaces including trees with geodesic convexity [6]. However, the general conjecture has recently been disproved by Bukh [1].

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The notion of a k -a.r set can be generalized to multi-sets by considering partitions of multi-sets rather than sets. In this paper, we define $\tilde{r}_k(\mathcal{C})$ to be the size of the largest k -a.r. multi-set. Note that $\tilde{r}_k(\mathcal{C}) \geq r_k(\mathcal{C}) - 1$, with equality for $k = 2$ (as a 2-a.r. multi-set is a set, i.e. no element can appear more than once). When $k = 2$, we often omit the prefix k , e.g. a 2-a.r. set may be called an a.r. set and $\tilde{r}(\mathcal{C}) = \tilde{r}_2(\mathcal{C})$.

In this paper we shall study P_3 -convexity in trees. For a graph G , a set U of vertices of G is P_3 -convex or, briefly, *convex* if every vertex not in U has at most one neighbour in U . Equivalently, U is convex if it contains all middle vertices in the paths of length 2 between two vertices of U . P_3 -convexity was first considered in the context of directed graphs and tournaments (see [4],[5],[7],[10]).

Throughout this paper graphs are always finite, simple and undirected. For a graph G , let $\tilde{r}_k(G)$ denote the k th Radon number for P_3 -convexity on G , and for a set $U \subseteq V(G)$, let $H_G(U)$ denote the convex hull of U in G .

As the first main result of our paper, we show that Eckhoff's conjecture holds for P_3 -convexity on trees.

Theorem 1.1. *Let T be a tree, $k \geq 3$. Then $\tilde{r}_k(T) \leq (k - 1)\tilde{r}_2(T)$.*

Given a graph G , call a set $A \subseteq V(G)$ *free* if every vertex of G has at most one neighbour in A . Note that every free set in a graph G is also convex and the converse does not hold in general. Let $\tilde{\alpha}(G)$ be the size of the largest free set in G . It follows that $\tilde{r}(G) \geq \tilde{\alpha}(G)$. Our second main theorem answers a question posed by Dourado et al. [2].

Theorem 1.2. *Let T be a tree. Then $\tilde{r}_2(T) \leq 2\tilde{\alpha}(T)$.*

We shall show that this theorem is sharp in the sense that there are infinitely many trees for which we have equality.

The last inequality is not true in general as shown by the graph G_1 in Figure 1. Every two of the seven vertices of G_1 have a common neighbour, hence $\tilde{\alpha}(G_1) = 1$. It is easy to check that the set $A = \{2, 4, 6\}$ of vertices of G_1 is a.r. and that every set of 4 vertices of G_1 is not a.r., therefore $\tilde{r}(G_1) = 3$.

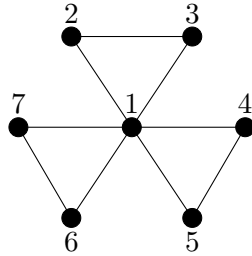


Figure 1: Graph G_1

We prove Theorem 1.1 in Section 2 and Theorem 1.2 in Section 3.

2 Proof of Theorem 1.1

Before proving this theorem, we introduce some notation. For a graph G and a vertex $v \in V(G)$, define

$$\tilde{r}_k^*(G, v) = \max\{|R| : R \text{ is a } k\text{-a.r. multi-set and } v \notin H_G(R)\}. \quad (1)$$

Proof of Theorem 1.1. We shall prove more than claimed in the statement of the theorem. Namely, we shall show that for every tree T

- $\tilde{r}_k^*(T, v) \leq (k-1)\tilde{r}_2^*(T, v)$ holds for every $v \in V(T)$,
- $\tilde{r}_k(T) \leq (k-1)\tilde{r}_2(T)$.

Our proof is by induction on $n = |V(T)|$. Both statements are clear for $n \leq 2$.

Let T be a tree with $n \geq 3$ vertices. The second statement follows easily by induction using expression 2 for \tilde{r}_k^* below. For a vertex $v \in V(T)$, let v_1, \dots, v_l be its neighbours, and for every $i \in [l]$ let T_i be the component of v_i in $T \setminus \{v\}$. Then

$$\tilde{r}_k^*(T, v) = \max_{i \in [l]} \left(\sum_{j \neq i} \tilde{r}_k^*(T_j, v_j) + \tilde{r}_k(T_i) \right). \quad (2)$$

We now prove that $\tilde{r}_k(T) \leq (k-1)\tilde{r}_2(T)$. Let R be a k -a.r. multi-set of maximal size. If T has endvertex v which is not in R , let $T' = T \setminus \{v\}$. Then by induction,

$$|R| = \tilde{r}_k(T') \leq (k-1)r_2(T') \leq (k-1)r_2(T).$$

Thus we may assume that R contains each endvertex of T at least once.

In the rest of the proof we consider two possible cases which will be dealt with in different subsections.

2.1 Case 1.

There is a longest path v_1, \dots, v_m in T such that $\deg(v_2) \geq 3$.

Let $z = v_3$, $y = v_2$ and x_1, \dots, x_l be the neighbours of y other than z . Note $l \geq 2$, and by the choice of v_1, \dots, v_m as a longest path, x_1, \dots, x_l are all endvertices (see Figure 2).

Denote by s_i , $i \in [l]$, the number of appearances of x_i in R and by t the number of appearances of y in R . By our assumption that R contains every endvertex at least once, $s_i \geq 1$ for every $i \in [l]$. As R is k -a.r., $s_i, t \leq k-1$. Let $s = s_1 + \dots + s_l$. We consider three cases according to the value of s .

(a) $s \leq 2k-2$.

Let $\tau = \min\{s, 2k-2-s\}$, $\sigma = (s-\tau)/2$. Note that σ is an integer and $\tau + \sigma \leq k-1$. Set $T' = T \setminus \{x_1, \dots, x_l\}$ (see Figure 2). Let R' be the multi-set obtained by adding σ copies of y to $R \cap V(T')$. Note $|R| = |R'| + s - \sigma = |R'| + \sigma + \tau \leq |R'| + k-1$.

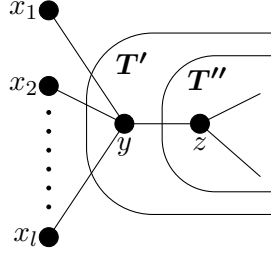


Figure 2: Case 1

Claim 2.1. R' is k -a.r..

Proof. We shall show the existence of sequences $a_1, \dots, a_\sigma, b_1, \dots, b_\sigma$ satisfying

- $a_j, b_j \in [l]$ and $a_j \neq b_j$ for every $j \in [\sigma]$,
- $|\{j \in [\sigma] : a_j = i\}| + |\{j \in [\sigma] : b_j = i\}| = s_i$ for every $i \in [l]$.

Note that the existence of such sequences completes the proof of this claim. Assume to the contrary that $R' = R'_1 \cup \dots \cup R'_k$ is a k -Radon partition of R' . Obtain R_1, \dots, R_k by replacing each of the σ new copies of y with a distinct pair x_{a_j}, x_{b_j} , $j \in [\sigma]$. By the choice of the a_j 's and b_j 's, $R = R_1 \cup \dots \cup R_k$ is a partition of R . Clearly $H_T(R_l) \cap V(T') = H_{T'}(R'_l)$ for every $l \in [k]$, hence this partition is a k -Radon partition of R , contrary to the choice of R as a k -a.r. set.

It thus remains to show the existence of such sequences. By induction on k , we show that if $s = s_1 + \dots + s_l \leq 2k - 2$ and $s_i \leq k - 1$ for every $i \in [l]$ we can find two sequences satisfying the above. We assume $\sigma \geq 1$ or equivalently $s \geq k$, because otherwise there is nothing to prove. When $k = 2$ we thus have that without loss of generality $s_1 = s_2 = 1$, and we set $a_1 = 1, b_1 = 2$. If $k \geq 3$ assume that $s_1 \geq s_2 \geq \dots \geq s_l$, and let $a_\sigma = 1, b_\sigma = 2$. Now set

$$s'_i = \begin{cases} s_i - 1 & i \in \{1, 2\} \\ s_i & \text{otherwise} \end{cases}.$$

Note that $s' = s'_1 + \dots + s'_l \leq 2k - 4$ and $s'_i \leq k - 2$ (otherwise $s_3 \geq k - 1$ and $s_1 + s_2 + s_3 \geq 3(k - 1) > 2(k - 1)$, a contradiction). Also $\sigma' = 2k - 4 - s' = \sigma - 1$. We can now continue by induction. \square

Using Claim 2.1 we conclude by induction that

$$\tilde{r}_k(T) = |R| \leq |R'| + (k - 1) \leq \tilde{r}_k(T') + (k - 1) \leq (k - 1)(\tilde{r}_2(T') + 1).$$

The following claim completes the proof of Theorem 1.1 in Case 1a.

Claim 2.2. $\tilde{r}_2(T) \geq r_2(T') + 1$.

Proof. Let S' be an a.r. set in T' of maximal size. Set

$$S = \begin{cases} S' \cup \{x_1\} & y \notin S' \\ (S' \setminus \{y\}) \cup \{x_1, x_2\} & y \in S' \end{cases}.$$

Note that $|S| = |S'| + 1$. We shall show that S is a.r., thus proving the claim. Assume to the contrary that there exists a partition $S = A \cup B$ with $H_T(A) \cap H_T(B) \neq \emptyset$. We assume $x_1 \in A$. Consider the following three possibilities.

- $y \notin S'$.

Set $A' = A \setminus \{x_1\}$. Note

- $S' = A' \cup B$ is a partition of S' .

$$- H_T(A) = \begin{cases} H_{T'}(A') \cup \{x_1\} & z \notin H_{T'}(A') \\ H_{T'}(A') \cup \{x_1, y\} & z \in H_{T'}(A') \end{cases}.$$

- $H_T(B) = H_{T'}(B)$, and $y \notin H_T(B)$.

Therefore $H_T(A) \cap H_T(B) = H_{T'}(A') \cap H_{T'}(B) = \emptyset$, a contradiction.

- $y \in S'$ and $x_1, x_2 \in A$.

Let $A' = (A \setminus \{x_1, x_2\}) \cup \{y\}$. Then $H_T(A) = H_{T'}(A') \cup \{x_1, x_2\}$, and $H_T(B) = H_{T'}(B)$.

As before we reach a contradiction.

- $y \in S'$ and $x_1 \in A, x_2 \in B$.

As S' is a.r. and $y \in S', z \notin H_{T'}(A \setminus \{x_1\}) \cap H_{T'}(B \setminus \{x_2\})$. Without loss of generality, $z \notin H_{T'}(B \setminus \{x_2\})$. Set $A' = (A \setminus \{x_1\}) \cup \{y\}$. As before $H_T(A) \subseteq H_{T'}(A') \cup \{x_1, y\}$ and $H_T(B) = H_{T'}(B) \cup \{x_2\}$. This leads to a contradiction to S' being a.r.

□

(b) $s = 2k - 1$.

Define T' as before, and let R' be the union of $R \cap V(T')$ with a copy of x_1 and $k - 1$ copies of y .

Claim 2.3. R' is k -a.r..

Proof. Replacing s_1 by $s_1 - 1$ returns us to the setting of Claim 2.1. Following the same arguments we obtain this claim. □

Set $T'' = T' \setminus \{y\}$, $R'' = R' \cap V(T'')$ (see Figure 2). Then $z \notin H_{T''}(R'')$ as otherwise we can partition R' into k parts, $k - 1$ of which contain y , and the last contains both x_1 and z . This partition is such that y is in the convex hull of all parts, contradicting the fact that R' is k -a.r.. Thus

$$\tilde{r}_k(T) = |R| = 2k - 1 + |R''| < 3(k - 1) + \tilde{r}_k^*(T'', z) \leq (k - 1)(3 + \tilde{r}_2^*(T'', z)).$$

The following claim completes the proof of Theorem 1.1 in Case 1b.

Claim 2.4. $\tilde{r}_2(T) \geq 3 + \tilde{r}_2^*(T'', z)$.

Proof. Let S'' be an a.r. set of T'' satisfying $z \notin H_{T''}(S'')$. We shall show that $S = S'' \cup \{x_1, x_2, x_3\}$ is a.r. thus proving the claim (note that $l \geq 3$, so S is well defined). Assume that we have a Radon partition $S = A \cup B$. Without loss of generality $x_1, x_2 \in A$. Set $A' = (A \cap V(T')) \cup \{y\}$, $B' = B \cap V(T')$. Then $H_T(A) \cap H_T(B) = H_{T'}(A') \cap H_{T'}(B')$. Claim 2.4 follows from the following claim.

Claim 2.5. Let T be a tree, $v \in V(T)$ and S an a.r. set in T satisfying $v \notin H_T(S)$. Let $T_{v \leftarrow u}$ denote the tree obtained from T by adding a new vertex u and connecting it to v . Then $S \cup \{u\}$ is a.r. in $T_{v \leftarrow u}$.

Proof. We prove the claim by induction on $|V(T)|$. The claim is clear when T has at most one vertex. Let $T' = T_{v \leftarrow u}$ and $S = A \cup B$ a partition of S . We show $H_{T'}(A \cup \{u\}) \cap H_{T'}(B) = \emptyset$.

Let v_1, \dots, v_l be the neighbours of v in T . Denote by T_i the component of v_i in $T \setminus \{v\}$ and $S_i = S \cap V(T_i)$, $A_i = A \cap V(T_i)$, $B_i = B \cap V(T_i)$. If for every $i \in [l]$, $v_i \notin H_{T_i}(A_i)$, then $H_{T'}(A \cup \{u\}) = H_{T'}(A) \cup \{u\}$ and $H_{T'}(B) = H_T(B)$. Thus, as S is a.r., $H_{T'}(A \cup \{u\}) \cap H_{T'}(B) = H_T(A) \cap H_T(B) = \emptyset$.

So we can assume $v_1 \in H_{T_1}(A_1)$. As $v \notin H_T(S)$, this means that $v_i \notin H_{T_i}(S_i)$ for every $i \geq 2$. As S is a.r., $v_1 \notin H_{T_1}(B_1)$. Thus

$$\begin{aligned} H_{T'}(A \cup \{u\}) &= H_{T_1}(A_1) \cup \{u\} \cup \left(\bigcup_{j \geq 2} H_{(T_j)_{v_i \leftarrow v}}(A_i \cup \{v\}) \right) \\ H_{T'}(B) &= \bigcup_{i \geq 1} H_{T_i}(B_i) = H_{T_1}(B_1) \cup \left(\bigcup_{i \geq 2} H_{(T_i)_{v_i \leftarrow v}}(B_i) \right). \end{aligned}$$

Therefore

$$H_{T'}(A \cup \{u\}) \cap H_{T'}(B) = \bigcup_{i \geq 2} (H_{(T_i)_{v_i \leftarrow v}}(A_i \cup \{v\}) \cap H_{(T_i)_{v_i \leftarrow v}}(B_i))$$

The proof follows using the induction hypothesis with T_i , $i \geq 2$. □

□

(c) $s \geq 2k$.

Similarly to Claim 2.1, we can conclude that the multi-set obtained by adding k copies of y to $R \cap V(T')$ is k -a.r., which is obviously a contradiction.

2.2 Case 2.

In every longest path v_1, \dots, v_m of T , $\deg(v_2) = 2$.

Fix a longest path v_1, \dots, v_m in T . Denote $v_3 = z$, and note that each of its neighbours other than v_4 is either an endvertex or has degree 2 and is adjacent to an endvertex (by the choice of the longest path and the definition of Case 2). Let y_1, \dots, y_p be the neighbours of z other than v_4 which have degree 2, and y_{p+1}, \dots, y_q the neighbours of z other than v_4 which are endvertices. Let x_1, \dots, x_p be the neighbours of y_1, \dots, y_p which are endvertices respectively (see Figure 3a).

Denote by s_i , $i \in [p]$, the number of appearances of x_i in R ; t_i , $i \in [q]$, the number of appearances of y_i in R and u the number of appearances of z in R . Let $t = t_1 + \dots + t_q$. As in the previous case, we conclude from the fact that R is k -a.r. that $t \leq 2k - 1$. Consider the following three cases.

(a) $q = 1$.

Then $t_1 + \min\{s_1, u\} \leq k - 1$ (otherwise obtain a k -Radon partition of R by putting a copy of y_1 in t_1 sets, and a copy of x_1 and z in the other $k - t_1$ sets). Thus

$$s_1 + t_1 + u = t_1 + \min\{s_1, u\} + \max\{s_1, u\} \leq 2(k - 1).$$

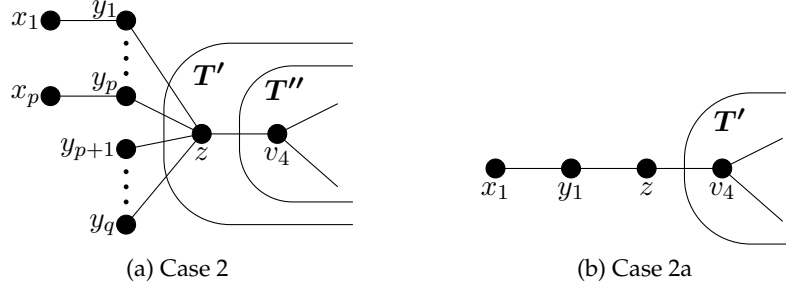


Figure 3: Case 2

Let $T' = T \setminus \{x_1, y_1, z\}$, $R' = R \cap V(T')$ (see Figure 3b). Then

$$\tilde{r}_k(T) = |R| \leq |R'| + 2(k-1) \leq \tilde{r}_k(T') + 2(k-1) \leq (k-1)(\tilde{r}_2(T') + 2).$$

The proof of Theorem 1.1 in Case 2a follows from the following claim.

Claim 2.6. $\tilde{r}_2(T) \geq \tilde{r}_2(T') + 2$.

Proof. Let S' be an a.r. set in T' . Set $S = S' \cup \{x_1, y_1\}$. It is easy to verify that S is a.r. □

(b) $t \leq 2k - 2$.

As before, set $\tau = \min\{t, 2k - 2 - t\}$, $\sigma = (t - \tau)/2$ (then $t - \sigma \leq k - 1$). Let $T' = T \setminus \{x_1, \dots, x_p, y_1, \dots, y_q\}$ (see Figure 3a) and let R' be the multi-set obtained by adding σ copies of z to $R' \cap V(T')$. Then as in Claim 2.1, R' is k -a.r. and thus

$$\tilde{r}_k(T) = |R| = |R'| + s + t - \sigma \leq \tilde{r}_k(T') + (p+1)(k-1) \leq (k-1)(\tilde{r}_2(T') + p+1).$$

The proof of Theorem 1.1 in this case follows from the following claim.

Claim 2.7. $\tilde{r}_2(T) \geq \tilde{r}_2(T') + p + 1$.

Proof. Let S' be a.r. in T' . Set

$$S = \begin{cases} S' \cup \{x_1, \dots, x_p, y_1\} & z \notin S' \\ (S' \setminus \{z\}) \cup \{x_1, \dots, x_p, y_1, y_2\} & z \in S' \end{cases}$$

Showing that S is a.r. is similar to the proof of Claim 2.2. □

(c) $t = 2k - 1$.

Let $T' = T \setminus \{x_1, \dots, x_p, y_1, \dots, y_q\}$, $T'' = T' \setminus \{z\}$ (see Figure 3a). Let $R'' = R \cap V(T'')$ and let R' be the multi-set obtained by adding $k-1$ copies of z to R'' and a copy of y_1 . As in Case 1b, R' is k -a.r. in T' and R'' is k -a.r. in T'' with $v_4 \notin H_{T''}(R'')$. Thus

$$\tilde{r}_k(T) = |R''| + s + 2k - 1 \leq \tilde{r}_k^*(T'', v_4) + s + 2k - 1 \leq (k-1)\tilde{r}_2^*(T'', v_4) + s + 2k - 1. \quad (3)$$

As $s_i \leq k-1$ for every $i \in [p]$, $s \leq (k-1)p$. If $s < (k-1)p$, we obtain

$$\tilde{r}_k(T) \leq (k-1)(\tilde{r}_2^*(T'', v_4) + p + 2).$$

And the proof of Theorem 1.1 in this case follows from the claim below.

Claim 2.8. $\tilde{r}_2(T) \geq \tilde{r}_2(T'', v_4) + p + 2$.

Proof. If S'' is a.r. in T'' with $v_4 \notin S''$ then similarly to the proof of Claim 2.4, $S = S'' \cup \{x_1, \dots, x_p, y_1, y_2\}$ is a.r. in T . \square

Thus we may assume $s = (k - 1)p$ i.e. $s_1 = \dots = s_p = k - 1$.

Claim 2.9. $t_1 = \dots = t_p = 0$.

Proof. Assume otherwise, then without loss of generality $t_1 \geq 1$. Let $\phi = k - t_1$. Similarly to the proof of Claim 2.1 we will show the existence of $a_1, \dots, a_\phi, b_1, \dots, b_\phi$ such that

- $a_j, b_j \in [2, q]$ and $a_j \neq b_j$ for every $j \in [\phi]$,
- $|\{j \in [\phi] : a_j = i\}| + |\{j \in [\phi] : b_j = i\}| \leq t_i$ for every $i \in [2, q]$.

This leads to a contradiction as we can then obtain a k -Radon partition of R by putting a copy of y_1 in t_1 of the sets, and putting a copy of x_1 and a pair y_{a_i}, y_{b_i} in each of the other $k - t_1$ sets. y_1 will be in the intersection of the convex hulls of the sets (here we use the assumption that $s_1 = k - 1$ so this is indeed possible).

If $t_i \leq k - t_1 - 1$ for every $i \in [2, q]$, we proceed as in Claim 2.1 to prove the existence of such sequences. Otherwise, let i_0 be such that $t_{i_0} \geq k - t_1$. Note that

$$\sum_{j \neq 1, i_0} t_j = t - t_{i_0} - t_1 \geq 2k - 1 - (k - 1) - t_1 = k - t_1.$$

Thus in this case we can choose $a_1 = \dots = a_\phi = i_0$ and $b_1, \dots, b_\phi \in [2, q] \setminus \{i_0\}$ to satisfy the requirements. \square

Using Claim 2.9 it follows that $2k - 1 = t = t_{p+1} + \dots + t_q$. As $t_i \leq k - 1$ for every $i \in [q]$, $q - p \geq 3$.

Claim 2.10. $\tilde{r}_2(T) \geq \tilde{r}_2^*(T'', v_4) + 3 + p$.

Proof. Let S'' be an a.r. set in T'' with $v_4 \notin H_{T''}(S'')$. Let $S = S'' \cup \{x_1, \dots, x_p, y_{p+1}, y_{p+2}, y_{p+3}\}$. It is easy to see that S is a.r. in T . \square

Recalling inequality 3, we obtain

$$\tilde{r}_k(T) \leq s + 2k - 1 + (k - 1)\tilde{r}_2^*(T'', v_4) \leq (k - 1)(p + 3 + \tilde{r}_2^*(T'', v_4)) \leq (k - 1)\tilde{r}_2(T).$$

And the proof of Theorem 1.1 is complete. \square

3 Proof of Theorem 1.2

We need the following definition for the proof of Theorem 1.2. Let G be a graph, $v \in V(G)$. Define

$$\tilde{\alpha}^*(T, v) \triangleq \max\{|A| : A \text{ is free and } v \notin A\}.$$

Proof of Theorem 1.2. We prove a stronger statement than what is claimed in this theorem. We shall show that for every tree T

- $\tilde{r}^*(T, v) \leq 2\tilde{\alpha}^*(T, v)$ for every vertex $v \in V(T)$,
- $\tilde{r}(T) \leq 2\tilde{\alpha}(T)$.

We prove these statements by induction on $n = |V(T)|$. Both statements are clear for $n \leq 3$.

To prove the first statement, let $v \in V(T)$, and denote by v_1, \dots, v_l its neighbours. For every $i \in [l]$, let T_i be the connected component of v_i in $T \setminus \{v\}$. It is easy to see that

$$\tilde{\alpha}^*(T, v) = \max_{j \in [k]} \left\{ \sum_{i \neq j} \tilde{\alpha}^*(T_i, v_i) + \tilde{\alpha}(T_j) \right\}.$$

Note the similarity to expression 2 from the previous section. It thus follows by induction that $\tilde{r}^*(T, v) \leq 2\tilde{\alpha}^*(T, v)$.

We now proceed to proving that $\tilde{r}(T) \leq 2\tilde{\alpha}(T)$. Let R be an a.r. set of maximal size in T . As in the proof of Theorem 1.1, we can assume that R contains all endvertices of T . The *brothers* of an endvertex v are the endvertices in distance 2 from v . Then in particular, every endvertex has at most 2 brothers, as no vertex of T can have more than 3 neighbours in R .

The following claim will be useful in the rest of the proof.

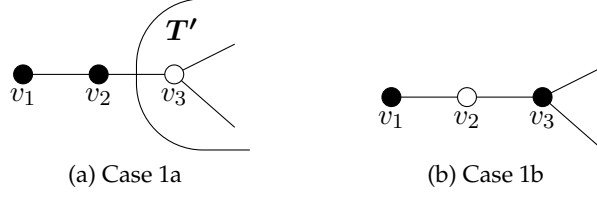
Claim 3.1. *Let T be a tree. There exists a free set $A \subseteq V(T)$ of size $\tilde{\alpha}(T)$ satisfying that for every endvertex $v \in V(T)$ either $v \in A$ or one of its brothers is in A .*

Proof. Let A be a free set in T of maximal size, v an endvertex in T and u its only neighbour. If $v \in A$ we are done. Otherwise, by the maximality of A , $A \cup \{v\}$ is not free. As u is the only neighbour of v , there is a neighbour $w \neq v$ of u which is contained in A . If w is an endvertex, we are done. Otherwise, set $A' = (A \setminus \{w\}) \cup \{v\}$. Then A' contains v and is free of size $\tilde{\alpha}(T)$. Continuing similarly will result in a free set of size $\tilde{\alpha}(T)$ with the property that for each endvertex either it or one of its brothers is in the set. \square

We consider three cases concerning longest paths in T . Note that the theorem can be easily verified if the longest path in T has at most 3 vertices, thus we assume that a longest path in T contains at least 4 vertices. We devote a separate subsection for each case.

3.1 Case 1.

There is a longest path v_1, \dots, v_m such that the component of v_4 in $T \setminus \{v_5\}$ has no endvertex in distance 3 from v_4 with brothers.



In this figure and the following ones:
a black vertex is in R , a white one is not in R ,
and for a grey vertex it is unknown if it is in R .

Figure 4: Cases 1a, 1b

We consider six cases.

(a) $v_1, v_2 \in R$.

Set $T' = T \setminus \{v_1, v_2\}$ (see Figure 4a), $R' = R \cap V(T')$.

R' is a.r. in T' and $v_3 \notin H_{T'}(R')$. Thus, by induction,

$$\tilde{r}(T) = |R'| + 2 \leq \tilde{r}^*(T', v_3) + 2 \leq 2(\tilde{\alpha}^*(T', v_3) + 1).$$

Note that $\tilde{\alpha}^*(T', v_3) + 1 \leq \tilde{\alpha}(T)$, because if $A' \subseteq V(T') \setminus \{v_3\}$ is free, then $A' \cup \{v_1\}$ is free in T . Therefore $\tilde{r}(T) \leq 2\tilde{\alpha}(T)$ in this case.

(b) $v_1, v_3 \in R$ (see Figure 4b).

Set $R' = (R \setminus \{v_3\}) \cup \{v_2\}$. It is easy to see that R' is a.r. and it follows from Case 1a that $\tilde{r}(T) \leq 2\tilde{\alpha}(T)$.

We can now assume that the above two cases do not occur. Consider the neighbours of v_3 other than v_4 . Each such neighbour either is an endvertex, or has degree 2 and its other neighbour is an endvertex (using the fact v_1, \dots, v_m is a longest path and that we are in Case 1). Let $S_i, i \in \{1, 2\}$, be the set of neighbours of v_3 other than v_4 with degree i . Note that $v_2 \in S_2$, and by our previous assumptions: $S_1 \subseteq R, S_2 \cap R = \emptyset$. In particular $|S_1| \leq 3$. Consider the remaining four cases.

(c) $|S_2| \geq 2$.

Let T' be the component of v_3 in $T \setminus (S_2 \setminus \{v_2\})$ (see Figure 5a). Then

$$\tilde{r}(T') \geq \tilde{r}(T) - (|S_2| - 1),$$

as $R \cap V(T')$ is a.r. in T' and $R \setminus V(T')$ contains only the endvertices which are neighbours of vertices in $S_2 \setminus \{v_2\}$. Furthermore

$$\tilde{\alpha}(T) \geq \tilde{\alpha}(T') + (|S_2| - 1).$$

To see this, let $A' \subseteq V(T')$ be a maximum sized free set in T' containing v_1 (recall Claim 3.1). Then $v_3 \notin A'$ and the set obtained by adding the endvertices which are neighbours of the vertices in S_2 to A' is free in T . Hence, by induction,

$$\tilde{r}(T) \leq \tilde{r}(T') + |S_2| - 1 < 2(\tilde{\alpha}(T') + |S_2| - 1) \leq 2\tilde{\alpha}(T).$$

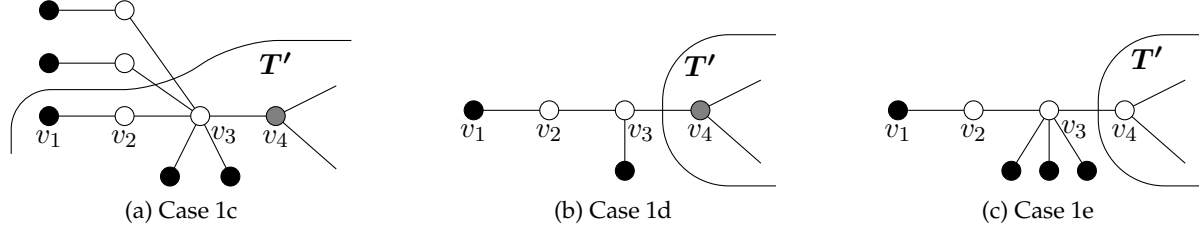


Figure 5: Case 1c, 1d, 1e

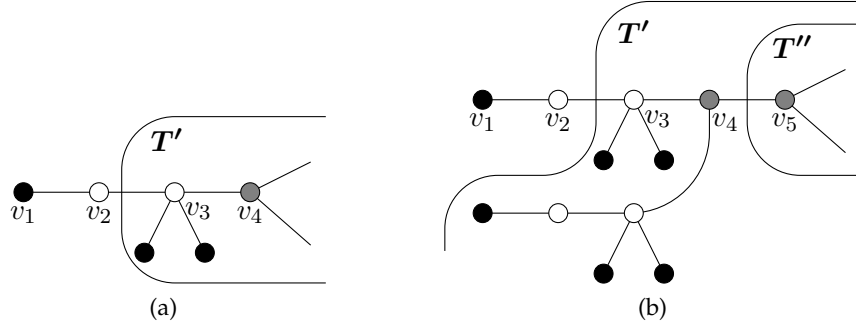


Figure 6: Case 1f

We can now assume that $S_2 = \{v_2\}$.

(d) $|S_1| \leq 1$.

Let T' be the connected component of v_4 in $T \setminus \{v_3\}$ (see Figure 5b). Then $\tilde{r}(T') \geq \tilde{r}(T) - 2$, as $R \cap V(T')$ is a.r. in T' , and $v_2, v_3 \notin R$ (otherwise consider Cases 1a, 1b). Also $\tilde{\alpha}(T) \geq \tilde{\alpha}(T') + 1$, because if $A' \subseteq V(T')$ is free in T' then $A' \cup \{v_1\}$ is free in T . We obtain

$$\tilde{r}(T) \leq \tilde{r}(T') + 2 \leq 2(\tilde{\alpha}(T') + 1) \leq 2\tilde{\alpha}(T).$$

(e) $|S_1| = 3$.

Let T' be as in the previous case (see Figure 5c) and set $R' = R \cap V(T')$. Then $v_4 \notin H_{T'}(R')$ and $|R| = |R'| + 4$. Also $\tilde{\alpha}(T) \geq \tilde{\alpha}^*(T, v_4) + 2$, because if $A' \subseteq V(T') \setminus \{v_4\}$ is free, then $A' \cup \{v_1, v_2\}$ is free in T . Hence

$$\tilde{r}(T) \leq \tilde{r}^*(T', v_4) + 4 \leq 2(\tilde{\alpha}^*(T, v_4) + 2) \leq 2\tilde{\alpha}(T).$$

(f) $|S_1| = 2$.

Set $T' = T \setminus \{v_1, v_2\}$ (see Figure 6a). If T' contains a free set of maximal size A' such that $v_3 \notin A'$, then $A' \cup \{v_1\}$ is free, so in this case $\tilde{\alpha}(T) \geq 1 + \tilde{\alpha}(T')$ and

$$\tilde{r}(T) \leq 1 + \tilde{r}(T') < 2(1 + \tilde{\alpha}(T')) \leq 2\tilde{\alpha}(T).$$

Therefore we may assume that v_3 is contained in every maximum sized free set of T' . Let S be the set of neighbours of v_4 other than v_5 and for $v \in S$ let T_v be the connected component of v in $T \setminus \{v_4\}$.

We need the following claim.

Claim 3.2. T_v has depth 2 as a tree rooted in v for every $v \in S$.

Proof. Let $v \in S$, $v \neq v_3$ (note that the claim is clear if $v = v_3$). By the choice of v_1, \dots, v_m as a longest path in T , T_v has depth at most 2. We now show that T_v has depth at least 2, i.e. v has neighbours which are not endvertices. Let A' be a free set of maximal size in T' . If v has no neighbour in T_v which is not an endvertex, then $(A' \setminus \{v_3\}) \cup \{v\}$ is also a free set of the same size in T' , contradicting our previous assumption. \square

If for some $v \in S$, T_v is not isomorphic to T_{v_3} (as rooted trees at v, v_3 respectively), by changing the selected longest path to go through v instead of v_3 , we go back to one of the previous cases. Thus we may assume that the trees T_v , $v \in S$, are all isomorphic to T_{v_3} .

Let T'' be the component of v_5 in $T \setminus \{v_4\}$ (see Figure 6b), $R'' = R \cap V(T'')$. Then $|R \setminus R''| \leq 3|S| + 1$ as for each $v \in S$, $|R \cap V(T_v)| = 3$ and possibly v_4 in R . Note also that $\tilde{\alpha}(T) \geq \tilde{\alpha}(T'') + 2|S|$, because the union of a free set of T'' with the endvertices in distance 3 from v_4 and their neighbours is free in T . Therefore

$$\tilde{r}(T) \leq \tilde{r}(T'') + 3|S| + 1 \leq 2\tilde{\alpha}(T'') + 4|S| \leq 2\tilde{\alpha}(T).$$

3.2 Case 2.

Case 1 does not hold, and there exists a longest path v_1, \dots, v_m such that the component of v_4 in $T \setminus \{v_5\}$ has no endvertices in distance 4 from v_4 with more than one brother.

Choose the longest path such that v_1 has exactly one brother v'_1 . Then $v_1, v'_1 \in R$ and as R is a.r., $v_2 \notin R$. We consider the neighbours of v_3 other than v_4 . Note that they can be of degrees 1, 2 or 3 only and that if they have degree 2 or 3 the other neighbours are endvertices. Let S_i , $i \in \{1, 2, 3\}$, be the set of neighbours of v_3 other than v_4 with degree i . Consider the following six cases.

(a) $S_2 \neq \emptyset$.

Let T' be the component of v_3 in $T \setminus S_2$ (i.e. remove all neighbours of v_3 of degree 2, see Figure 7a). Then, by induction

$$\tilde{r}(T) \leq \tilde{r}(T') + 2|S_2| \leq 2(\tilde{\alpha}(T') + |S_2|) \leq 2\tilde{\alpha}(T).$$

The last inequality follows from the fact that a maximum free set in T' can be assumed to contain v_1 (see Claim 3.1), so it does not contain v_3 and we can add the $|S_2|$ endvertices that were discarded to obtain a free set in T .

We now assume $S_2 = \emptyset$.

(b) $|S_3| \geq 2$.

Set T' to be the component of v_3 in $T \setminus (S_3 \setminus \{v_3\})$ (see Figure 7b). R contains $2(|S_3| - 1)$ of the discarded vertices, thus, as in the previous case,

$$\tilde{r}(T) \leq \tilde{r}(T') + 2(|S_3| - 1) \leq 2(\tilde{\alpha}(T') + (|S_3| - 1)) \leq 2\tilde{\alpha}(T).$$

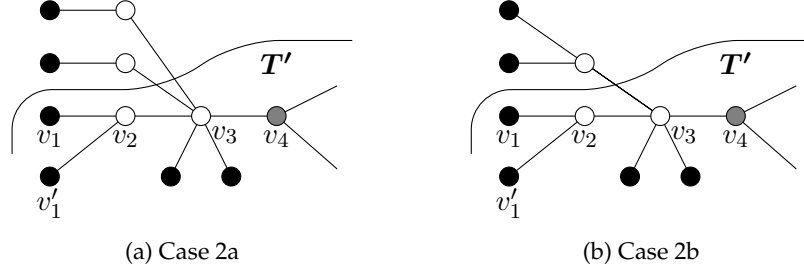


Figure 7: Cases 2a, 2b

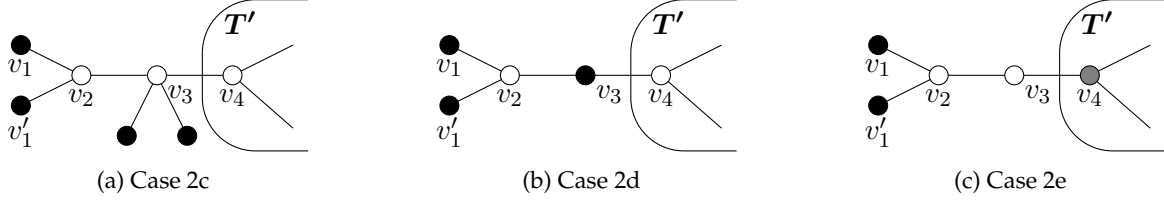


Figure 8: Cases 2c, 2d, 2e

Hence we can assume $|S_3| = 1$. Note that $|S_1| \leq 2$, since R is a.r..

(c) $|S_1| = 2$.

Set T' the component of v_4 in $T \setminus \{v_3\}$ (see Figure 8a). Then

$$\tilde{r}(T) \leq 4 + \tilde{r}^*(T', v_4) \leq 2(2 + \tilde{\alpha}^*(T', v_4)) \leq 2\tilde{\alpha}(T),$$

since $v_4 \notin H_{T'}(R \cap V(T'))$ (R is a.r.) and the union of a free set in $V(T') \setminus \{v_4\}$ with $\{v_1, v_2\}$ remains free.

(d) $S_1 = \emptyset$ and $v_3 \in R$.

Choose T' as in the previous case (see Figure 8b). Again $v_4 \notin H_{T'}(R \cap V(T'))$ and similarly

$$\tilde{r}(T) \leq 3 + \tilde{r}^*(T', v_4) < 2(2 + \tilde{\alpha}^*(T', v_4)) \leq 2\tilde{\alpha}(T).$$

(e) $S_1 = \emptyset$ and $v_3 \notin R$.

Again set T' as before (see Figure 8c). Here R contains only 2 of the discarded vertices and we can add v_1 to any free set of T' to obtain a free set of T . Thus

$$\tilde{r}(T) \leq \tilde{r}(T') + 2 \leq 2(\tilde{\alpha}(T') + 1) \leq 2\tilde{\alpha}(T).$$

(f) All previous cases do not hold, i.e. $|S_1| = |S_3| = 1$ and $S_2 = \emptyset$.

Set $T' = T \setminus \{v_1, v_1', v_2\}$ (see Figure 9a). If T' contains a maximum free set A' with $v_3 \notin A'$, then $A' \cup \{v_1\}$ is free in T and thus

$$\tilde{r}(T) \leq \tilde{r}(T') + 2 \leq 2(1 + \tilde{\alpha}(T')) \leq 2\tilde{\alpha}(T).$$

Therefore, we may assume that every maximum free set of T' contains v_3 . As in Case 1f, let S be the set of neighbours of v_4 different from v_5 and for $v \in S$ define T_v to be the component of v in $T \setminus \{v_4\}$. Similarly to Claim 3.2, T_v has depth 2 as a tree rooted at v for every $v \in S$. If T_v is

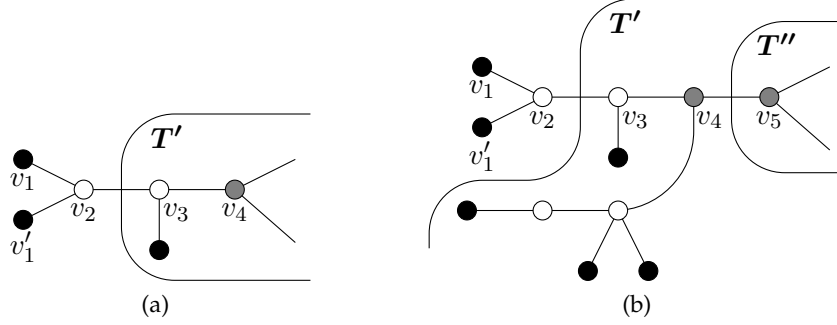


Figure 9: Case 2f

not isomorphic to T_{v_3} or to the graph in Case 1f (as rooted trees), by changing the longest path to go through v , we can continue as before. (Note that in all but the present case and Case 1f, we did not consider other neighbours of v_4).

Set T'' to be the component of v_5 in $T \setminus \{v_4\}$ (see Figure 9b). R contains three vertices of T_v for every $v \in S$ and possibly it contains v_4 as well. Also $\tilde{\alpha}(T) \geq \tilde{\alpha}(T'') + 2|S|$, as we can two vertices from each T_v to a free set of T'' to obtain a free set. Thus

$$\tilde{r}(T) \leq \tilde{r}(T'') + 3|S| + 1 \leq 2(\tilde{\alpha}(T'') + 2|S|) \leq 2\tilde{\alpha}(T).$$

3.3 Case 3.

For every choice of a longest path v_1, \dots, v_m the connected component of v_4 in $T \setminus \{v_5\}$ has an endvertex with 2 brothers in distance 3 from v_4 .

We choose the longest path such that v_1 has 2 brothers v'_1 and v''_1 . Then $v_1, v'_1, v''_1 \in R$, so $v_2, v_3 \notin R$.

Similarly to Cases 2a, 2b, we can assume that all the neighbours of v_3 are endvertices (except for maybe v_4) having one or two endvertices as neighbours. By the choice of v_1, \dots, v_m as a longest path, the neighbours of v_3 other than v_4 are either endvertices or have degree 4 and are neighbours to 3 endvertices. Thus, as R is a.r., v_3 can have degree 2 or 3 only. Set T' to be the component of v_4 in $T \setminus \{v_3\}$. We consider seven possible cases.

- (a) v_3 has degree 3 with the only neighbour other than v_2 and v_4 being an endvertex (see Figure 10a).

Then $v_4 \notin H_{T'}(R \cap V(T'))$. Note that $\tilde{\alpha}(T) \geq \tilde{\alpha}^*(T', v_4) + 2$, as we can add v_1 and v_2 to a free set in $V(T') \setminus \{v_4\}$ to obtain a free set. Thus

$$\tilde{r}(T) \leq \tilde{r}^*(T', v_4) + 4 \leq 2(\tilde{\alpha}^*(T', v_4) + 2) \leq 2\tilde{\alpha}(T).$$

- (b) v_3 has degree 3 with the only neighbour other than v_2 and v_4 , u , having three neighbours which are endvertices (see Figure 10b).

Then again $v_4 \notin H_{T'}(R \cap V(T'))$, and $\tilde{\alpha}(T) \geq 3 + \tilde{\alpha}^*(T', v_4)$, as we can add v_1 , v_2 and an endvertex which is a neighbour of u to a free set of T' . Thus

$$\tilde{r}(T) \leq 6 + \tilde{r}^*(T', v_4) \leq 2(3 + \tilde{\alpha}^*(T', v_4)) \leq 2\tilde{\alpha}^*(T).$$

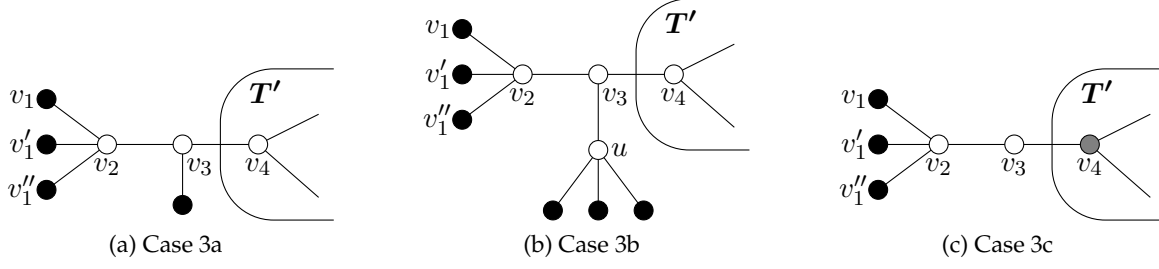


Figure 10: Cases 3a, 3b, 3c

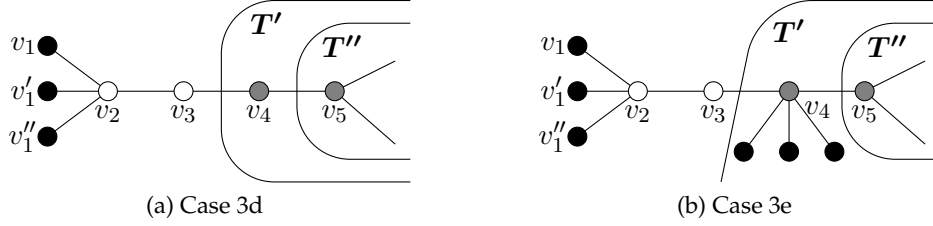


Figure 11: Cases 3d, 3e

In the remaining cases we assume that v_3 has degree 2. Let T'' be the component of v_5 in $T \setminus \{v_4\}$.

- (c) T' has a maximum free set A' with $v_4 \notin A'$ (see Figure 10c).
Then $A' \cup \{v_1, v_2\}$ is free and

$$\tilde{r}(T) \leq 3 + \tilde{r}(T') < 2(\tilde{\alpha}(T') + 2) \leq 2\tilde{\alpha}(T).$$

We may now assume that every maximum free set of T' contains v_4 . Let S be the set of neighbours of v_4 other than v_3 and v_5 .

Claim 3.3. *The vertices in S are endvertices in T .*

Proof. Let $v \in S$, and T_v the component of v in $T \setminus \{v_4\}$. Then T_v has depth at most 2 as a tree rooted in v (by the choice of v_1, \dots, v_m as a longest path). Let A' be a free set of maximal size in T' , then $v_4 \in A'$. If v has a neighbour in T_v , u , it is either an endvertex, or all of its neighbours except for v are endvertices. Then $(A' \setminus \{v_4\}) \cup \{u\}$ is free in T' , a contradiction. Thus v has no neighbours in T_v , i.e. it is an endvertex in T . \square

Clearly, the claim implies $|S| \leq 3$.

- (d) $S = \emptyset$ (see Figure 11a).

Then $|R \cap (V(T) \setminus V(T''))| \leq 4$ and we can add v_1, v_2 to a free set of T'' . Thus

$$\tilde{r}(T) \leq 4 + \tilde{r}(T'') \leq 2(2 + \tilde{\alpha}(T'')) \leq 2\tilde{\alpha}(T).$$

We can assume now that $S \neq \emptyset$.

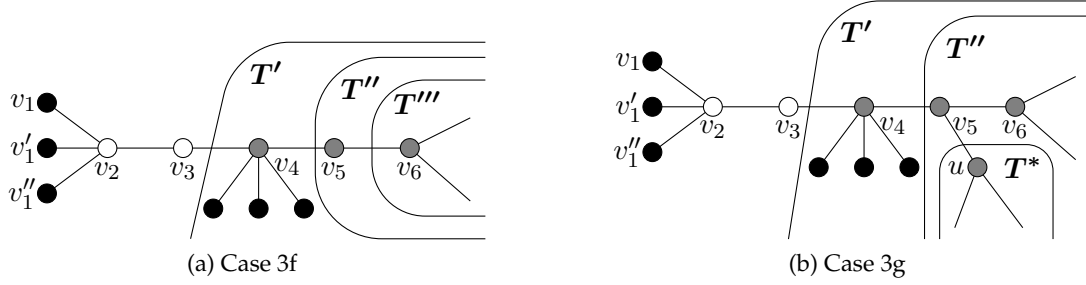


Figure 12: Cases 3f, 3g

- (e) There is a free set A'' of maximal size in T'' with $v_5 \notin A''$ (see Figure 11b). Then $|R \cap (V(T) \setminus V(T''))| \leq 6$ and the union of A'' with v_1, v_2 and an endvertex from S is free, thus

$$\tilde{r}(T) \leq 6 + \tilde{r}(T'') \leq 2(3 + \tilde{\alpha}(T'')) \leq 2\tilde{\alpha}(T).$$

Thus we can assume that every maximal free set in T'' contains v_5 .

- (f) v_5 has degree 2 in T .

Let T''' be the component of v_6 in $T \setminus \{v_5\}$ (see Figure 12a). Then, as in the previous case,

$$\tilde{r}(T) \leq 6 + \tilde{r}(T''') \leq 2(3 + \tilde{\alpha}(T''')) \leq 2\tilde{\alpha}(T).$$

- (g) v_5 has a neighbour $u \neq v_4, v_6$.

Let T^* be the component of u in $T \setminus \{v_5\}$ (see Figure 12b). The following claim can be proved similarly to the proofs of Claims 3.2, 3.3, using the above assumptions.

Claim 3.4. T^* has depth 3 as a tree rooted in u .

By considering a longest path going through u instead of v_4 , we can assume that the component of u in $T \setminus \{v_5\}$ satisfies the same conditions as the component of v_4 . However, in this case T'' has an endvertex in distance 2 from v_5 , a contradiction to the assumption that every maximum free set of T'' contains v_5 .

□

The following example shows that Theorem 1.2 is sharp.

This is sequence of trees T_m , $m \geq 1$, with $10m$ vertices. $\tilde{r}(T) \geq 6m$ (the set of all endvertices is a.r.). Let A be a free set of T_m with maximal size. We can assume that A contains the endvertices $x_1, \dots, x_m, y_1, \dots, y_m$. Thus $u_1, \dots, u_m \notin A$. Also, A contains at most one of the 3 neighbours of u_i for each $i \in [m]$. Hence $\{x_1, \dots, x_m\} \cup \{y_1, \dots, y_m\} \cup \{w_1, \dots, w_m\}$ is a free set of maximal size, so $\tilde{\alpha}(T) = 3m$. By Theorem 1.2, $\tilde{r}(T) \leq 2\tilde{\alpha}(T) = 6m$. Thus $\tilde{r}(T) = 6m = 2\tilde{\alpha}(T)$.

4 Concluding Remarks

In this paper we proved two results about the Radon number for P_3 -convexity in graphs. It may be interesting to consider these problems for general graphs. Regarding Theorem 1.1, it is still an

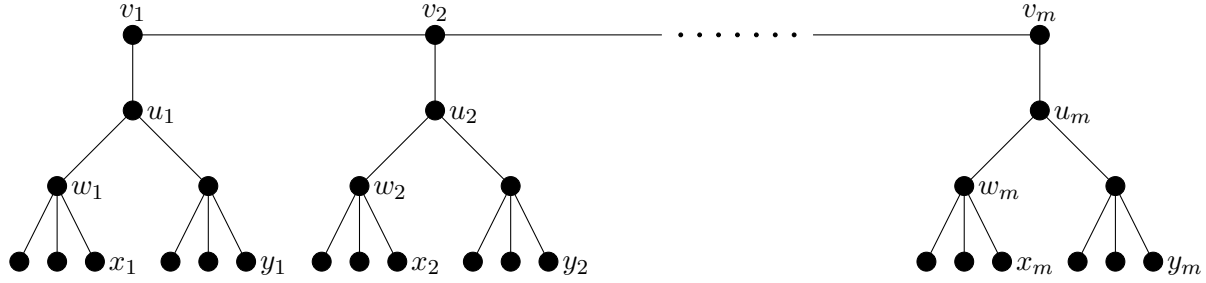


Figure 13: Sharpness of Theorem 1.2

open problem to determine whether Eckhoff's conjecture holds for P_3 -convexity in all graphs. We showed that the inequality $\tilde{r}(G) \leq 2\tilde{\alpha}(G)$ from Theorem 1.2, does not hold for all graphs G , but it may still be the case that a similar but weaker inequality holds in general. Furthermore, for both results, it would be interesting to characterize the trees for which the results hold with equality.

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